A PARABOLIC ANALOGUE OF THE HIGHER-ORDER COMPARISON THEOREM OF DE SILVA AND SAVIN

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ABSTRACT. We show that the quotient of two caloric functions which vanish on a portion of the lateral boundary of a $H^{k+\alpha}$ domain is $H^{k+\alpha}$ up to the boundary for $k \geq 2$. In the case $k = 1$, we show that the quotient is in $H^{1+\alpha}$ if the domain is assumed to be space-time $C^{1,\alpha}$ regular. This can be thought of as a parabolic analogue of a recent important result in [DS1], and we closely follow the ideas in that paper. We also give counterexamples to the fact that analogous results are not true at points on the parabolic boundary which are not on the lateral boundary, i.e., points which are at the corner and base of the parabolic boundary.

1. INTRODUCTION

The classical comparison theorem states that two nonnegative harmonic functions which vanish on the boundary of a Lipschitz, or more in general a NTA domain, must vanish at the same rate. An important consequence of this result is that the quotient of two such functions is, in fact, Hölder continuous up to the boundary (only the function in the denominator needs now to be nonnegative now). In some recent remarkable works De Silva and Savin have established a higher-order version of such result. Specifically, they have proved in [DS1] the following.

Theorem 1.1. *Let D be a $C^{k,\alpha}$ domain in $\mathbb{R}^n$, with $0 \in \partial D$. Let u, v be two harmonic functions vanishing on $\partial D \cap B(0,1)$. Furthermore, $u > 0$ in D and $u = 1$ at some interior point in D. Then,*

$$\left\|\frac{v}{u}\right\|_{C^{k,\alpha}(B(0,1/2))} \leq C||v||_{L^\infty(B(0,1))}. \tag{1.1}$$

The classical Schauder estimates imply that u, v are $C^{k,\alpha}$ up to the boundary. Then, by the Hopf Lemma we have $u_\nu > 0$, and from this one can assert that the quotient $\frac{v}{u}$ is $C^{k-1,\alpha}$ up to the boundary. However, Theorem 1.1 remarkably states that the ratio is in fact $C^{k,\alpha}$ up to the boundary. The case $k = 0$ of this result is the *boundary Harnack principle* mentioned in the opening, see [CFMS] and [JK].

The purpose of this note is to generalize Theorem 1.1 above to the heat equation and, more generally, to linear parabolic equations with variable coefficients. The main results are Theorem 3.1 and Theorem 4.5 below. Although our work has been strongly motivated by that of De Silva and Savin, it has nonetheless required some delicate adaptations to the parabolic setting.

It is worth mentioning here that, besides being an interesting regularity result in its own right, a direct application of Theorem 1.1 above implies C^∞ smoothness of a priori $C^{1,\alpha}$ free boundaries without the use of the hodograph transformation as in [KN], [KNS], a tool that so far has been the standard way of establishing smoothness of free boundaries starting from $C^{1,\alpha}$. For this aspect one should see Corollary 1.2 in [DS1]. Having said this, we would like to mention that the hodograph transformation in [KN], [KNS] does in fact imply real-analyticity of the free boundary, which is instead not implied by Theorem 1.1. Nevertheless, Theorem 1.1 provides a

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new perspective in the study of Schauder theory and free boundary problems. Theorem 1.1 has also been extended to slit domains in [DS2]. In the same paper such result has been used to establish smoothness of the free boundary in lower-dimensional obstacle problems of Signorini type near regular points. The real analyticity of the free boundary near regular points in the elliptic thin obstacle problem has been recently established in [KPS] by using a method based on hodograph transformation.

These recent results and their applications to free boundary problems motivated us to investigate their parabolic counterpart. Our main result is Theorem 3.1 below which constitutes the heat equation counterpart of Theorem 1.1. We mention that the case $k = 0$ of Theorem 3.1 can be found in [FSY]. In the present paper we make the observation that the ideas in [DS1] can be successfully adapted to the parabolic situation. The idea of the proof in [DS1] is to approximate (after a suitable change of coordinates) v by polynomials of the type x_nP by a compactness argument which uses Schauder estimates. Then, finish by remarkable idea that x_nP can in fact be replaced by uP. In our situation, as in Theorems 3.1 and 4.5, we show that the approximating polynomials in the space variable in [DS1] can be suitably replaced by appropriate approximating parabolic polynomials, and one can argue in a similar manner. Modulo some delicate details, which we have tried to illustrate as much as possible. Similarly to Corollary 1.2 in [DS1], Theorem 3.1 implies, in particular, the C^∞ smoothness in the parabolic obstacle problem of a $C^{1,\alpha}$ free boundary near the regular points considered in Theorem 13.1 in [CPS]. We note nonetheless that, analogously to the elliptic case, one can establish the space-like real analyticity of the free boundary by employing the hodograph transform as in [CPS]. It remains to be seen whether the analogue of Theorem 1.1 in [DS2] is true for parabolic equations since such result would have important applications to the parabolic thin obstacle problem which was systematically studied in [DGPT]. This question will be addressed in a future study.

This paper is organized as follows. To better demonstrate the ideas we first establish in Section 2 the higher-regularity result in the case $k = 1$ and for the heat equation. In Section 3, still for the heat equation, we analyze the case $k \geq 2$. In Section 4 we extend the results of the previous sections to non-divergence form operators with variable coefficients. Section 5 closes the paper. In it we present an application of Theorems 2.2 and 3.1 to the parabolic obstacle problem studied in [CPS].

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2. $H^{1+\alpha}$ regularity for the heat equation

In this section we establish the case $k = 1$ of the main higher regularity result which is Theorem 3.1 below. Our main result is Theorem 2.2. In order to state it we need to introduce some preliminary notation and hypothesis.

With $x \in \mathbb{R}^n, t \in \mathbb{R}$, we will denote by (x,t) a generic point in $\mathbb{R}^{n+1}$. If $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$ we indicate with

$$Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^2, t_0] \tag{2.1}$$

the parabolic cylinder "centered" at (x_0,t_0). Given an open set $G \subset \mathbb{R}^{n+1}$ we say that a point $(x_0,t_0) \in \partial G$ belongs to the parabolic boundary of G, and write $(x_0,t_0) \in \partial_p G$, if for every $r > 0$ the open cylinder $B_r(x_0) \times (t_0 - r^2, t_0)$ contains points of the complement of G (notice that $(x_0,t_0) \notin B_r(x_0) \times (t_0 - r^2, t_0)$). Thus, for instance, when $G = \Omega \times (0,T)$, then $\partial_p G = (\Omega \times \{0\}) \bigcup (\partial\Omega \times [0,T])$. We denote by SG the lateral boundary of G, see p. 13 of [Li] for the relevant notions. The reader should also see p. 5 of [Li] for the notion of parabolic norms and distance. For the for the definitions of $C^{k,\alpha}$ spaces, norm and seminorm, we refer the reader

to p. 90 in [GT]. We also refer to p. 46 in [Li] for the relevant notion of $H^{k+\alpha}$ spaces and p. 75 in [Li] as well for the definition of domains with $H^{k+\alpha}$ boundaries.

We now consider a connected bounded open set $G \subset \{(x,t) \in \mathbb{R}^{n+1} \mid t \le 1\}$, and we assume that $(0,0) \in \partial_p G$. We also suppose that $\partial G \cap Q_2(0,0) = SG \cap Q_2(0,0)$, and that $G \cap Q_2(0,0)$ is space-time $C^{1,\alpha}$ regular, i.e., there exists $f \in C^{1,\alpha}$ such that

$$G \cap Q_2(0,0) = \{(x,t) \in Q_2(0,0) \mid x_n > f(x',t)\}.$$

If we introduce the following notations:

$$F_t = \{x \in \mathbb{R}^n \mid (x,t) \in G \cap Q_2(0,0)\}, \quad G_t = \{x \in \mathbb{R}^n \mid (x,t) \in \partial G \cap Q_2(0,0)\},$$

then for each $t \le 0$, the set G_t is a $(n-1)$-dimensional $C^{1,\alpha}$ submanifold which can be equivalently characterized in the following manner

$$G_t = \{(x', x_n) \mid x_n = f(x',t) \text{ and } (x,t) \in Q_2(0,0)\}.$$

For each $x \in G_t$ we denote by $\nu_t(x)$ the inward unit normal to F_t at x. In the following discussion, whenever there i son ambiguity about the point (x,t) that is being considered, we will simply write ν instead of $\nu_t(x)$.

We will use the notation $D'f$ when referring to the gradient of f with respect to the variable $x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}$. Without loss of generality, we may and will assume that

$$\nu = \nu_0(0) = e_n, \quad f(0,0) = 0, \quad D'f(0,0) = 0, \text{ and } ||f||_{C^{1,\alpha}} \le c_0, \tag{2.2}$$

where c_0 is a dimensional constant which is chosen sufficiently small in such a way that $(e_n, -3/2) \in G$. This latter property can always be achieved by suitably scaling the domain as we describe later. We also normalize the function u appearing in Theorem 2.2 below so that the following holds

$$u(e_n, -3/2) = 1. \tag{2.3}$$

Remark 2.1. *Hereafter in this paper when we say that a constant is* universal *we mean that it depends only on the dimension n, and the parameters α and k in Theorem 3.1 below. We notice that in the next Theorem 2.2 we are taking $k = 1$, and thus in this case the dependence would be only on n and α.*

The main result of this section is the following $H^{1+\alpha}$ regularity result.

Theorem 2.2. *Let the domain $G \subset \mathbb{R}^{n+1}$ be as above and satisfy the assumption* (2.2)*. Let u and v be two solutions to the heat equation in $G \cap Q_2(0,0)$, with u, v vanishing on $\partial_p G \cap Q_2(0,0)$, and suppose that $u > 0$ in $G \cap Q_2(0,0)$ and that it satisfy the normalization* (2.3)*. Then, for some universal $C > 0$ one has*

$$||\frac{v}{u}||_{H^{1+\alpha}(G\cap Q_1(0,0))} \le C\big(||v||_{L^\infty(G\cap Q_2(0,0))} + 1\big). \tag{2.4}$$

Remark 2.3. *It is worth mentioning here that, although in Theorem 2.2 we assume that the domain be $C^{1,\alpha}$, instead of just $H^{1+\alpha}$ regular, it is not restrictive in its application to free boundary problems (See Corollary 5.2 below). This follows from the fact that the classical boundary Harnack inequalities imply that for the parabolic obstacle problem studied in* [CPS] *the Lipschitz free boundary near a regular point as in Theorem 13.1 in* [CPS] *is shown in the subsequent Theorem 14.1 in the same paper to be space-time $C^{1,\alpha}$ regular. The hypothesis of $C^{1,\alpha}$ regularity in Theorem 2.2 is only used to apply the Hopf Lemma as in Theorem 3' in* [LN]*. Note that the $H^{1+\alpha}$ regularity assumption on the domain does not imply that (4.6) in* [LN] *holds, which is precisely why we assume that the domain be $C^{1,\alpha}$ regular.*

Remark 2.4. *We now illustrate by an example in* [G] *that Theorem 2.2 cannot possibly be true at the base or at the corner points of a smooth cylinder. Consider to fix the ideas $G =$*

$B(0,1) \times (0,1)$. Let u, v be the solutions of the heat equation in G corresponding to Cauchy-Dirichlet data $g(x,t) = t^\alpha$ and $h(x,t) = t^\beta$ respectively. Clearly, u and v vanish at $t = 0$. Assume that $\beta < \alpha$ and denote by $K(x,t,y,s)$ the kernel function for G at the boundary such that

$$u(x,t) = \int_0^t \int_{\partial B} K(x,t,y,s) g(y,s) d\sigma(y) ds, \quad v(x,t) = \int_0^t \int_{\partial B} K(x,t,y,s) h(y,s) d\sigma(y) ds.$$

Then, we have

$$\frac{v(x,t)}{u(x,t)} = \frac{\int_0^t \int_{\partial B} K(x,t,y,s) s^\beta d\sigma(y) ds}{\int_0^t \int_{\partial B} K(x,t,y,s) s^\alpha d\sigma(y) ds} \geq \frac{1}{t^{\alpha-\beta}} \frac{\int_0^t \int_{\partial B} K(x,t,y,s) s^\alpha dy ds}{\int_0^t \int_{\partial B} K(x,t,y,s) s^\alpha dy ds} = \frac{1}{t^{\alpha-\beta}}.$$

We conclude that the ratio $\frac{v}{u}$ cannot possibly be bounded as $t \to 0^+$. This example demonstrates that Theorem 2.2 is not true in a neighborhood of a point $(x_0, 0) \in B \times \{0\}$.

The same example can be modified to demonstrate that Theorem 2.2 is not true in a neighborhood of a corner point $(x_0, 0) \in \partial B \times \{0\}$. Let ϕ be a smooth function on ∂B such that ϕ vanishes in a neighborhood of x_0, and let this time $g(x,t) = \phi(x) t^\alpha$, $h(x,t) = \phi(x) t^\beta$. As above, we obtain $\frac{v(x,t)}{u(x,t)} \geq \frac{1}{t^{\alpha-\beta}}$, and therefore the ratio $\frac{v}{u}$ is not bounded in a neighborhood of $(x_0, 0)$.

Before proving Theorem 2.2 we make some preliminary considerations and reductions, and we establish a crucial auxiliary lemma. With u, v as in Theorem 2.2, by Schauder theory (see for instance Theorem 4.27 in [Li]), we have that $u, v \in H^{1+\alpha}$ up to $\partial_p G \cap Q_{3/2}(0,0)$, say. Moreover, by the Hopf lemma in Theorem 3' in [LN], the Schauder type estimates, (2.2), the normalization condition (2.3) and the interior Harnack inequality for parabolic equations, we have that

$$u_\nu(x,t) \geq c > 0 \text{ in } \partial_p G \cap Q_1(0,0), \tag{2.5}$$

where, we recall, ν indicates the inward normal at $(x,t) \in D_t$. After parabolic dilations of the domain, i.e., by considering the rescaled functions

$$u_{r_0}(x,t) = \frac{u(r_0 x, r_0^2 t)}{r_0}, \qquad v_{r_0}(x,t) = \frac{v(r_0 x, r_0^2 t)}{r_0}, \tag{2.6}$$

we see that u_{r_0}, v_{r_0} solve the heat equation in the rescaled domain

$$G^{r_0} = \{(x,t) \in \mathbb{R}^{n+1} \mid (r_0 x, r_0^2 t) \in G\}.$$

Moreover, G^{r_0} is given near $(0,0)$ by the graph of $f_{r_0}(x',t) = \frac{f(r_0 x', r_0^2 t)}{r_0}$. From (2.6) it is easy to see that for every $(x,t), (y,s) \in G^{r_0} \cap Q_2(0,0)$ one as

$$\frac{|D_x u_{r_0}(x,t) - D_x u_{r_0}(y,s)|}{(|x-y|^2 + |t-s|)^{\alpha/2}} \leq r_0^\alpha [D_x u]_{H^\alpha(G \cap Q_2(0,0))}.$$

Therefore, given $\delta > 0$, we can choose the scaling parameter $r_0 = r_0([D_x u]_{H^\alpha(G \cap Q_2(0,0))}, \delta) > 0$ in such a way that

$$[D_x u_{r_0}]_{H^\alpha(G^{r_0} \cap Q_2(0,0))} \leq \delta. \tag{2.7}$$

It is then clear that for each $\delta > 0$ the corresponding function u_{r_0} satisfying (2.7) does depend on δ. We also note that because of (2.2), if for a given $\delta > 0$ the scaling parameter r_0 is suitably chosen, and if u is multiplied by an appropriate constant depending on c in (2.5), we can ensure that the following holds

$$||f_{r_0}||_{C^{1,\alpha}} \leq \delta, \ D_x u_{r_0}(0,0) = e_n. \tag{2.8}$$

We note in passing that the first inequality in (2.8) represents a flatness assumption of the boundary of G which will become important in establishing (2.32) below. Moreover, the choice of δ which is to be fixed later will be determined by Lemma 2.6 where a compactness argument, in which we let $\delta \to 0$, is employed. More precisely, in the proof of Lemma 2.6, δ is chosen small

enough so that (2.20) and (2.23) below hold for some $\rho > 0$ universal. In conclusion, given the choice of δ determined by Lemma 2.6, from now on to simplify the notation we will let $u_{r_0} = u$, $v_{r_0} = v$, $G^{r_0} = G$, and establish our results for these new u, v and G.

We now introduce the relevant notion of approximating affine function.

Definition 2.5. *We call P an approximating affine function at $(0,0)$ if it has the form $P(x) = \sum_{i=1}^n a_i x_i + a_0$, with $a_n = 0$.*

As in [DS1], we have the following intermediate lemma.

Lemma 2.6. *Assume that for some $r \leq 1$ and P an approximating affine function with $|a_i| \leq 1$, one has*

$$||v - uP||_{L^\infty(G\cap Q_r(0,0))} \leq r^{2+\alpha}. \tag{2.9}$$

Then, there exists an approximating affine function $\tilde{P}$ such that for some $C, \rho > 0$ universal, we have

$$||P - \tilde{P}||_{L^\infty(G\cap Q_r(0,0))} \leq C r^{1+\alpha}, \tag{2.10}$$

and

$$||v - u\tilde{P}||_{L^\infty(G\cap Q_{\rho r}(0,0))} \leq (\rho r)^{2+\alpha}. \tag{2.11}$$

Proof. We let $\tilde{G} = \{(\frac{x}{r}, \frac{t}{r^2}) \mid (x,t) \in G\}$, and consider the function $\tilde{v}$ defined in $\tilde{G}$ by the following equation

$$v(x,t) = u(x,t)P(x) + r^{2+\alpha}\tilde{v}(\frac{x}{r}, \frac{t}{r^2}). \tag{2.12}$$

Although in the next definition $\tilde{u}$ does not have the same meaning as $\tilde{v}$ in (2.12) above, for later purposes we nonetheless abuse the notation and set

$$\tilde{u}(x,t) = \frac{u(rx, r^2 t)}{r}, \qquad (x,t) \in \tilde{G}. \tag{2.13}$$

Before proceeding we pause to recall that in the reduction which precedes Lemma 2.6, given any $\delta > 0$ we have chosen $r_0 > 0$, depending on δ, such that the function $u = u_{r_0}$ satisfy (2.7) and (2.8). Since we also set $v = v_{r_0}$, it is clear that both u and v do depend on δ, and therefore so do $\tilde{u}$ and $\tilde{v}$. The reader should keep this in mind when below we let $\delta \to 0$ along a sequence.

Since u, v and P are solutions of the heat equation, we easily obtain from (2.12)

$$0 = (\Delta v - v_t)(x,t) = 2\sum_{i=1}^{n-1} a_i D_i u(x,t) + r^\alpha (\Delta \tilde{v} - \tilde{v}_t)(\frac{x}{r}, \frac{t}{r^2}), \tag{2.14}$$

where we have denoted by $Du = (D_1 u, ..., D_n u)$. Moreover, by (2.8) we know that $D_i u(0,0) = 0$ for $i = 1, ..., n-1$, and since by (2.7) we also have $[Du]_{H^\alpha(G\cap Q_2(0,0))} \leq \delta$, we conclude that for all $i = 0, ...n-1$,

$$||D_i u||_{L^\infty(G\cap Q_r(0,0))} \leq \delta r^\alpha. \tag{2.15}$$

Therefore, by using (2.15) in (2.14) we see that in $\tilde{G} \cap Q_1(0,0)$ one has

$$|\Delta \tilde{v} - \tilde{v}_t| \leq C\delta. \tag{2.16}$$

Furthermore, by (2.12) we obtain from (2.9) the following bound

$$||\tilde{v}||_{L^\infty(\tilde{G}\cap Q_1(0,0))} \leq 1.$$

In addition, $\tilde{v}$ vanishes on $\partial\tilde{G} \cap Q_1(0,0)$. Therefore, if we let $\delta \to 0$ along a sequence, we will have by compactness (by using uniform interior $H^{2+\alpha}$ estimates and boundary $H^{1+\alpha}$ estimates)

that for a subsequence $\delta \to 0$, we have $\tilde{v} = \tilde{v}(\delta) \to v_0$ uniformly on compact subsets, where the limit function v_0 has the properties

$$\begin{cases} \Delta v_0 - D_t v_0 = 0 \;\text{ in } B_1^+ \times (-1,0], \\ ||v_0||_{L^\infty} \leq 1, \\ v_0 = 0 \text{ on } (\{x_n = 0\} \cap B_1(0)) \times (-1,0]. \end{cases} \tag{2.17}$$

Here, $B_1^+ = B_1 \cap \{x_n > 0\}$. We now denote by V_0 the odd reflection in x_n of the function v_0 to the whole $B_1(0) \times (-1,0]$. Then, $\Delta V_0 - D_t V_0 = 0$ in $B_1(0) \times (-1,0]$, and V_0 satisfies the remaining two conditions in (2.17) above. In particular, from the smoothness of V_0 up to $(\{x_n = 0\} \cap B_1(0)) \times (-1,0]$ and the third property in (2.17) above we see that $D_i V_0 = 0$ for $i = 1,...,n-1$, and $D_t V_0 = 0$ on $(\{x_n = 0\} \cap B_1(0)) \times (-1,0]$. This gives $D_{ij} V_0 = 0$, $D_{jt} V_0 = 0$, $i,j = 1,...,n-1$, and $D_{tt} V_0 = 0$ in $(\{x_n = 0\} \cap B_1(0)) \times (-1,0]$. Furthermore, since V_0 is odd in x_n we also have $D_{nn} V_0(0,0) = 0$. Using the fact that the variable t has weight two, from the Taylor expansion of V_0 at $(0,0)$ we obtain that there exists $\rho > 0$ universal ($\rho \leq (4C)^{-1/(1-\alpha)}$ would do), such that

$$||v_0 - x_n Q_0||_{L^\infty(B_\rho^+(0)\times[-\rho^2,0])} \leq C\rho^3 \leq \frac{1}{4}\rho^{2+\alpha}, \tag{2.18}$$

where $Q_0(x) = \Sigma_{i=1}^{n-1} q_i x_i + q_0$ is an affine function which is approximating (note that $q_n = 0$ is a consequence of $D_{nn} V_0(0,0) = 0$), with

$$|q_i| \leq C_1 \tag{2.19}$$

for some $C_1 > 0$ universal. In particular, the product $x_n Q_0(x)$ is harmonic. We now fix such a universal $\rho > 0$. Since $\tilde{v} = \tilde{v}(\delta) \to v_0$ uniformly on compact sets as $\delta \to 0$ on a subsequence, by compactness we see that for δ sufficiently small we have,

$$||\tilde{v} - v_0||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))} \leq \frac{1}{4}\rho^{2+\alpha}. \tag{2.20}$$

From (2.20) and (2.18) we thus conclude that for δ sufficiently small

$$\begin{aligned} ||\tilde{v} - x_n Q_0||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))} &\leq ||\tilde{v} - v_0||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))} + ||v_0 - x_n Q_0||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))} \\ &\leq \frac{1}{4}\rho^{2+\alpha} + \frac{1}{4}\rho^{2+\alpha} = \frac{1}{2}\rho^{2+\alpha}. \end{aligned} \tag{2.21}$$

Moreover, from (2.7) and (2.8) we have that

$$||\tilde{u} - x_n||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))} \leq \delta. \tag{2.22}$$

Thus, from the triangle inequality, if with C_1 as in (2.19) we further restrict δ in dependence of ρ so that $nC_1\delta \leq \frac{1}{2}\rho^{2+\alpha}$, we have

$$||\tilde{v} - \tilde{u} Q_0||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))} \leq \rho^{2+\alpha}. \tag{2.23}$$

The conclusion now follows by taking $\tilde{P}(x) = P(x) + r^{1+\alpha} Q_0(\frac{x}{r})$ and by rewriting $\tilde{v}, \tilde{u}$ in terms of v, u. We notice explicitly that (2.10) follows from the definition of $\tilde{P}$ and from the fact that (2.19) gives

$$||Q_0(\frac{\cdot}{r})||_{L^\infty(G\cap Q_r(0,0))} \leq nC.$$

By using (2.12), (2.13) and the definition of $\tilde{P}$, we finally obtain

$$\begin{aligned} ||v - u\tilde{P}||_{L^\infty(G\cap Q_{\rho r}(0,0))} &= ||r^{2+\alpha}\tilde{v}(\frac{\cdot}{r}, \frac{\cdot}{r^2}) - r^{1+\alpha} u Q_0(\frac{\cdot}{r})||_{L^\infty(G\cap Q_{\rho r}(0,0))} \\ &= r^{2+\alpha}||\tilde{v}(\frac{\cdot}{r}, \frac{\cdot}{r^2}) - \tilde{u}(\frac{\cdot}{r}, \frac{\cdot}{r^2}) Q_0(\frac{\cdot}{r})||_{L^\infty(G\cap Q_{\rho r}(0,0))} \\ &\leq (\rho r)^{2+\alpha}, \end{aligned}$$

where in the last inequality we have used (2.23). This proves (2.11), thus completing the proof.

□

With this lemma in hand, one can establish Theorem 2.2 by arguing as in [DS1]. We nevertheless provide the details for the sake of completeness.

Proof of Theorem 2.2. Now suppose there exists an affine function P for which the hypothesis of Lemma 2.6 holds for some $r > 0$. Then with $\tilde{v}, \tilde{u}$ as in the proof of Lemma 2.6, we first note that by $H^{1+\alpha}$ estimates up to the boundary as in Theorem 4.27 in [Li] and from the fact that $\tilde{v}$ vanishes on $\tilde{G} \cap Q_2(0,0)$, we have for every $(x,t) \in \tilde{G} \cap Q_{1/2}(0,0)$

$$|\tilde{v}(x,t)| \leq C_1 d(x, \tilde{G}_t), \tag{2.24}$$

where $\tilde{G}_t = \{x \mid (x,t) \in \partial\tilde{G} \cap \overline{Q}_1(0,0)\}$, and $d(x,\tilde{G}_t)$ is the Euclidean distance of the point $x \in \mathbb{R}^n$ from $\tilde{G}_t$. Note that (2.24) is a reformulation of the Lipschitz estimate in the x variable at the boundary. Moreover, from (2.5) and the definition (2.13) of $\tilde{u}$ we find

$$\tilde{u}_\nu \geq c > 0,$$

where c is the universal constant in (2.5). From this inequality we easily obtain

$$\tilde{u}(x,t) \geq C_2 d(x, \tilde{G}_t). \tag{2.25}$$

The estimates (2.24) and (2.25) imply that

$$|\tilde{v}| \leq C\tilde{u} \qquad \text{in } \tilde{G} \cap Q_{1/2}(0,0). \tag{2.26}$$

We now see how the desired conclusion (2.4) above can be derived from (2.26) and from Lemma 2.6.

First, by rewriting $\tilde{v}, \tilde{u}$ in terms of v and u using (2.12), (2.13), we obtain as a direct consequence of (2.26)

$$|v - uP| \leq Cur^{1+\alpha} \text{ in } G \cap Q_{r/2}(0,0). \tag{2.27}$$

Moreover, since by (2.25) the function $\tilde{u}$ is bounded away from zero from below in $\tilde{G} \cap Q_{1/4}(\frac{1}{2}e_n, 0))$, from $H^{1+\alpha}$ estimates for $\tilde{v}$ and the fact that this function satisfies (2.16), we also have

$$||\frac{\tilde{v}}{\tilde{u}}||_{H^{1+\alpha}(\tilde{G} \cap Q_{1/4}(\frac{1}{2}e_n,0))} \leq C\big(||v||_{L^\infty(G \cap Q_2(0,0))} + 1\big). \tag{2.28}$$

In (2.28) we have used the fact that, because of (2.8) and the definition of $\tilde{G}$, at each time level $t \in (-1,0]$ the set $\tilde{K}_t = \{x \mid (x,t) \in \tilde{G} \cap Q_{1/4}(\frac{1}{2}e_n, 0)\}$ is at a Euclidean distance from $\tilde{G}_t$ which is bounded below by C_5, for some $C_5 > 0$ universal. In addition, the identity

$$\frac{v(x,t)}{u(x,t)} = P(x) + r^{1+\alpha} \frac{\tilde{v}(\frac{x}{r}, \frac{t}{r^2})}{\tilde{u}(\frac{x}{r}, \frac{t}{r^2})}, \tag{2.29}$$

which follows from (2.12), (2.13), implies that

$$[D_x(\frac{v}{u})]_{H^\alpha(G \cap Q_{r/4}(\frac{r}{2}e_n,0))} = [D_x(\frac{\tilde{v}}{\tilde{u}})]_{H^\alpha(\tilde{G} \cap Q_{1/4}(\frac{1}{2}e_n,0))} \tag{2.30}$$

and

$$< \frac{v}{u} >_{1+\alpha; G \cap Q_{r/4}(\frac{r}{2}e_n,0))} = < \frac{\tilde{v}}{\tilde{u}} >_{1+\alpha; \tilde{G} \cap Q_{1/4}(\frac{1}{2}e_n,0))} . \tag{2.31}$$

(See page 46 in [Li] for the definition of the seminorm $< w >_{1+\alpha;\Omega}$ which corresponds to the $\frac{1+\alpha}{2}$-Hölder seminorm of w in t).

Summing up, (2.28), (2.29), (2.30) and (2.31) imply the following estimate

$$||\frac{v}{u}||_{H^{1+\alpha}(G \cap Q_{r/4}(\frac{r}{2}e_n,0))} \leq C\big(||v||_{L^\infty(G \cap Q_2(0,0))} + 1\big). \tag{2.32}$$

We would like to mention that the region $G\cap Q_{r/4}(\frac{r}{2}e_n,0)$ is at a parabolic distance proportional to r from ∂G, a fact that follows from the first inequality in (2.8) which represents a flatness assumption on the boundary.

Multiplying v by a suitable constant, we may now assume that the hypothesis of the Lemma 2.6 is satisfied for r_0 small and $P=0$. Given any $0<r<1$, with ρ fixed as in Lemma 2.6 we choose $k\in\mathbb{N}$ such that $\rho^{k+1}r_0\le r<\rho^k r_0$. We thus apply Lemma 2.6 iteratively, i.e., first for r_0, then for $\rho r_0, \rho^2 r_0$ and so on. We finally obtain a limiting affine function P_0 such that

$$||v-uP_0||_{L^\infty(G\cap Q_r(0,0))}\le Cr^{2+\alpha} \text{ for } r\le r_0. \tag{2.33}$$

We note explicitly that

$$P_0(x)=\sum_{i=1}^{\infty}(\rho^{i-1}r_0)^{1+\alpha}Q_i(\frac{x}{\rho^{i-1}r_0}),$$

where Q_i is the affine function obtained after the i-th application of Lemma 2.6 in the iteration argument. Given that (2.33) holds with $P=P_0$ for every $r\le r_0$, therefore for any given $r\le r_0$, we can now repeat the arguments which lead to (2.27) with $P=P_0$ and consequently obtain for all $(x,t)\in G\cap Q_1(0,0)$

$$\left|\frac{v}{u}(x,t)-P_0(x)\right|\le C(|x|^2+|t|)^{\frac{1+\alpha}{2}}. \tag{2.34}$$

This implies the $H^{1+\alpha}$-regularity at the boundary point $(0,0)$. Combining (2.34) with (2.32), the desired conclusion follows by arguing for instance as in the proof of Proposition 4.13 in [CC]. We mention that, although the latter result is for the elliptic setting, the same argument goes through in the parabolic setting when the Euclidean distance in $\mathbb{R}^n$ is replaced by the parabolic distance in $\mathbb{R}^{n+1}$.

□

3. Higher regularity

In this section we establish the case $k>1$ of Theorem 2.2. Our main result is the following.

Theorem 3.1. *Let G be of class $H^{k+\alpha}$ in $G\cap Q_2(0,0)$, the other assumptions on G being as in Section 2. Let u and v be two solutions of the heat equation in $G\cap Q_2(0,0)$ such that u,v vanish on $\partial_p G\cap Q_2(0,0)$. Also, suppose that $u>0$ in $G\cap Q_2(0,0)$ and assume that it satisfy the normalization* (2.3). *Then, for some $C>0$ universal one has*

$$||\frac{v}{u}||_{H^{k+\alpha}(G\cap Q_1(0,0))}\le C\left(||v||_{L^\infty(G\cap Q_2(0,0))}+1\right). \tag{3.1}$$

Before proving the theorem above, we introduce some additional notations. Henceforth, we let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Given a multi-index $m=(m_1,....m_n)\in\mathbb{N}_0^n$, for $x=(x_1,...,x_n)\in\mathbb{R}^n$ we denote by $x^m=x_1^{m_1}\ ...\ x_n^{m_n}$ the monomial of degree $|m|=m_1+...+m_n$. Henceforth, for $i=1,...,n$, the notation $\overline{i}$ will indicate the multi-index in $\mathbb{N}_0^n$ which has 1 in the i-th position and 0 elsewhere.

Definition 3.2. *By a parabolic polynomial of degree $k\in\mathbb{N}_0$ we mean an expression of the form*

$$P(x,t)=\sum_{0\le|m|+2\ell\le k}a_{m,\ell}x^m t^\ell,$$

where $a_{m,\ell}\in\mathbb{R}$. Given such a P we define its norm as follows

$$||P||=\max_{0\le|m|+2\ell\le k}|a_{m,\ell}|.$$

Following [DS1], see also Chapter 1 in [CC], we introduce the following definition.

Definition 3.3. *We say that a function f is pointwise $H^{k+\alpha}$ at $(0,0)$ if there exists a parabolic polynomial $P(x,t)$ of degree k such that $f(x,t) = P(x,t) + O\left((|x|^2+|t|)^{\frac{k+\alpha}{2}})\right)$. We indicate this with $f \in H^{k+\alpha}(0,0)$ and we denote with $||f||_{H^{k+\alpha}(0,0)}$ the smallest $M > 0$ such that $||P|| \leq M$ and $|f(x,t) - P(x,t)| \leq M(|x|^2 + |t|)^{\frac{k+\alpha}{2}})$ for $t \leq 0$. We say that f is pointwise $H^{k+\alpha}$ at (x_0,t_0) if the function $h(x,t) = f(x+x_0, t+t_0)$ is pointwise $H^{k+\alpha}$ at $(0,0)$, and we indicate this with $f \in H^{k+\alpha}(x_0,t_0)$. Finally, given a bounded set $G \subset \mathbb{R}^{n+1}$ we say that $f \in H^{k+\alpha}(G)$ if*

$$||f||_{H^{k+\alpha}(G)} \overset{def}{=} \sup_{(x_0,t_0)\in G} ||f||_{H^{k+\alpha}(x_0,t_0)} < \infty.$$

We note that the class $H^{k+\alpha}(G)$ coincides with that defined on p. 46 in [Li].

Now with u, v as in Theorem 3.1, it follows from the Schauder theory (see Chapters 4 and 5 in [Li]) that u, v are in $H^{k+\alpha}$ up to $\overline{G \cap Q_2(0,0)}$. As in Section 2, we assume as well that

$$\nu = \nu_0(0) = e_n, \quad f(0,0) = 0, \; D'f(0,0) = 0, \text{ and } ||f||_{H^{k+\alpha}} \leq c_0, \tag{3.2}$$

where c_0 is a dimensional constant which is chosen sufficiently small in such a way that $(e_n, -3/2) \in G$. This latter property can always be achieved by suitably scaling the domain as in (2.6)-(2.8). Furthermore, we assume that

$$Du(0,0) = e_n, \quad ||u - x_n||_{H^{k+\alpha}(G\cap Q_2(0,0))} \leq \delta, \quad ||f||_{H^{k+\alpha}} \leq \delta. \tag{3.3}$$

The second inequality in (3.3) can be seen as follows. Since $u(0,0) = 0$ there is a parabolic polynomial P_u, of degree at most k, such that for all $(x,t) \in G \cap Q_{3/2}(0,0)$ one has

$$|u(x,t) - P_u(x,t)| \leq C(|x|^2 + |t|)^{\frac{k+\alpha}{2}}. \tag{3.4}$$

Note that such a polynomial corresponds to the weighted Taylor expansion of order k of u at $(0,0)$ in which derivatives in t are assigned the weighted order 2. If we let u_{r_0} be as in (2.6), then with $Du(0,0) = e_n$ we have that

$$|u_{r_0}(x,t) - x_n| \leq \frac{r_0^2}{r_0} |P_1(r_0 x, r_0^2 t)| = r_0 |P_1(r_0 x, r_0^2 t)|, \tag{3.5}$$

where $P_1(x,t) = P_u(x,t) - x_n$. Therefore, by choosing r_0 sufficiently small the second inequality in (3.3) can be ensured with our new $u = u_{r_0}$ and $G = G^{r_0}$, following computations similar to those in Section 2.

Let P be a given parabolic polynomial of degree k. By using the fact that $\Delta u - u_t = 0$, we have that

$$(\Delta - D_t)(uP) = 2 < Du, DP > + u(\Delta P - P_t). \tag{3.6}$$

As a first step we write a formula for (3.6) when $P(x,t) = x^m t^\ell$, for a given multi-index $m \in \mathbb{N}_0^n$ and $\ell \in \mathbb{N}_0$. This is (3.8) below. To derive such formula we first notice that from (3.3) it follows that $u = x_n + w$ where w is of parabolic order ≥ 2 and $||w||_{H^{k-1,\alpha}(G\cap Q_2(0,0))} \leq \delta$. On the other hand, (3.4) gives $u(x,t) = P_u(x,t) + z(x,t)$, where P_u is a polynomial of degree at most k. Combining these two facts, and letting $P_1(x,t) = P_u(x,t) - x_n$, we can thus write

$$u(x,t) = x_n + P_1(x,t) + z(x,t), \tag{3.7}$$

where, we note explicitly, the polynomial P_1 is of degree at least two, provided it is nonzero. We thus have

$$Du(x,t) = e_n + DP_1(x,t) + Dz(x,t).$$

This gives

$$\begin{aligned} 2 < Du, DP > + u(\Delta P - P_t) &= 2D_n P + x_n(\Delta P - P_t) + 2 < DP_1, DP > \\ &\quad + 2 < Dz, DP > + (P_1 + z)(\Delta P - P_t). \end{aligned}$$

Keeping in mind that

$$D_n P = m_n x^{m-\overline{n}} t^{\ell}, \quad x_n D_{nn} P = m_n(m_n-1) x^{m-\overline{n}} t^{\ell},$$

we find

$$\begin{aligned}(\Delta - D_t)(uP) &= m_n(m_n+1)x^{m-\overline{n}}t^{\ell} + \sum_{i\neq n} m_i(m_i-1)x^{m-2\overline{i}+\overline{n}}t^{\ell} - \ell x^{m+\overline{n}}t^{\ell-1} \\ &+ \left\{2 < DP_1, DP > + P_1(\Delta P - P_t)\right\} + \left\{2 < Dz, DP > + z(\Delta P - P_t)\right\}.\end{aligned}$$

It is now easy to see that

$$2 < DP_1, DP > + P_1(\Delta P - P_t) = \sum_{0\leq |q|+2\kappa \leq |m|+2\ell+k-2} c_{q,\kappa}^{m,\ell} x^q t^{\kappa}.$$

We note explicitly that, since P_1 is of degree at least 2, one has that $c_{q,\kappa}^{m,\ell} \neq 0$ only when $|m| + 2\ell \leq |q| + 2\kappa$. However, for later purposes we find it convenient to isolate from the right-hand side of the latter equation a parabolic polynomial which is of degree at most $k-1$, i.e., we decompose

$$\begin{aligned}\sum_{0\leq |q|+2\kappa \leq |m|+2\ell+k-2} c_{q,\kappa}^{m,\ell} x^q t^{\kappa} &= \sum_{|m|+2\ell\leq |q|+2\kappa \leq k-1} c_{q,\kappa}^{m,\ell} x^q t^{\kappa} \\ &+ \sum_{k\leq |q|+2\kappa \leq |m|+2\ell+k-2} c_{q,\kappa}^{m,\ell} x^q t^{\kappa}.\end{aligned}$$

If we define

$$w_{m,\ell}(x,t) = 2 < Dz, DP > + z(\Delta P - P_t) + \sum_{k\leq |q|+2\kappa \leq |m|+2\ell+k-2} c_{q,\kappa}^{m,\ell} x^q t^{\kappa},$$

then we conclude that

$$\begin{aligned}(\Delta - D_t)(uP) &= m_n(m_n+1)x^{m-\overline{n}}t^{\ell} + \sum_{i\neq n} m_i(m_i-1)x^{m-2\overline{i}+\overline{n}}t^{\ell} - \ell x^{m+\overline{n}}t^{\ell-1} \\ &+ \sum_{|m|+2\ell\leq |q|+2\kappa \leq k-1} c_{q,\kappa}^{m,\ell} x^q t^{\kappa} + w_{m,\ell}(x,t),\end{aligned} \tag{3.8}$$

where we have $|c_{q,\kappa}^{m,\ell}| \leq C\delta$, a fact which follows from $||w||_{H^{k-1,\alpha}(G\cap Q_2(0,0))} \leq \delta$. Moreover, again from (3.3) we have for all $(x,t) \in G \cap Q_1(0,0)$ that

$$|w_{m,\ell}(x,t)| \leq C\delta((|x|^2 + |t|)^{\frac{k-1+\alpha}{2}}). \tag{3.9}$$

In (3.9) we have crucially used the fact that $|Dz(x,t)| \leq C\delta(|x|^2 + |t|)^{\frac{k-1+\alpha}{2}})$ for all $(x,t) \in G \cap Q_1(0,0)$.

We now turn to determining a suitable expression for the right-hand side in (3.6) in the case in which

$$P(x,t) = \sum_{0\leq |m|+2\ell \leq k} a_{m,\ell} x^m t^{\ell}$$

is a general parabolic polynomial of degree k. In such case, we obtain from (3.8)

$$(\Delta - D_t)(uP) = R(x) + w(x), \tag{3.10}$$

where R is the parabolic polynomial of degree $k-1$ given by

(3.11)

$$R(x,t) = \sum_{0\le|q|+2\kappa\le k-1} d_{q,\kappa}x^q t^\kappa = \sum_{0\le|m|+2\ell\le k} a_{m,\ell}\Big\{ m_n(m_n+1)x^{m-\bar{n}}t^\ell + \sum_{i\neq n} m_i(m_i-1)x^{m-2\bar{i}+\bar{n}}t^\ell - \ell x^{m+\bar{n}}t^{\ell-1} + \sum_{|m|+2\ell\le|q|+2\kappa\le k-1} c_{q,\kappa}^{m,\ell}x^q t^\kappa \Big\},$$

and

$$w(x) = \sum_{0\le|m|+2\ell\le k} a_{m,\ell}w_{m,\ell}(x).$$

Note that (3.11) gives

$$(q_n+1)(q_n+2)a_{q+\bar{n},\kappa} + \sum_{i\neq n}(q_i+1)(q_i+2)a_{q+2\bar{i}-\bar{n},\kappa} - (\kappa+1)a_{q-\bar{n},\kappa+1} + c_{q,\kappa}^{m,\ell}a_{m,\ell} = d_{q,\kappa}. \tag{3.12}$$

Given a parabolic polynomial of degree $k-1$ such as $R(x) = \sum_{0\le|q|+2\kappa\le k-1} d_{q,\kappa}x^q t^\kappa$, our objective is finding a parabolic polynomial of degree k, $P(x,t) = \sum_{0\le|m|+2\ell\le k} a_{m,\ell}x^m t^\ell$, such that (3.10) hold (in particular, we will be interested in this section in the case when $R\equiv 0$, see Definition 3.4 below). This will be possible if, given constants $d_{q,\kappa}$, we can solve the linear system (3.12) above for the unknowns $a_{m,\ell}$. Notice that one can think of (3.12) as an equation where $a_{q+\bar{n},\kappa}$ is a linear combination of $d_{q,\kappa}$'s and $a_{m,\ell}$'s such that either $|m|+2\ell<|q|+2\kappa+1$, or when $|m|+2\ell = q+2\kappa+1$, then $m_n < q_n+1$. It thus follows that we can solve (3.12) if we arbitrarily assign all the coefficients $a_{m,\ell}$ when $m=(m_1,...,m_n)$ is a multi-index having $m_n=0$. In this respect we emphasize that the crucial fact which makes this claim possible is that the third term in the left-hand side of (3.12), namely, the one which comes from differentiating in the time variable t, has the same degree as the first two terms. We define the order of a coefficient $a_{m,\ell}$ as $|m|+2\ell$, i.e., the weighted degree of the corresponding monomial $x^m t^\ell$.

To verify the above claim we briefly describe the procedure of determining the coefficients. First of all, $a_{(0,...,0),0}$, which is the unique coefficient of order 0, is assigned arbitrarily since it trivially satisfies the requirement $m_n=0$. We proceed by a double induction. Suppose we know all coefficients $a_{m,\ell}$ up to order p. Given a coefficient a_{m_0,ℓ_0} of order $p+1$, i.e., $|m_0|+2\ell_0 = p+1$, let $(m_0)_n$ denote the entry at the n-th position of the corresponding multi-index (m_0,ℓ_0). Clearly, $0\le(m_0)_n\le p+1$. We determine all coefficients of order $p+1$ by induction on $(m_0)_n$. First, all coefficients a_{m_0,ℓ_0} of order $p+1$ with $(m_0)_n=0$ are arbitrarily assigned. Suppose now all coefficients a_{m_0,ℓ_0} of order $p+1$ with $(m_0)_n\le\chi$ are known. Then, given a coefficient a_{m_1,ℓ_1} of order $p+1$ with $(m_1)_n=\chi+1$, we have from (3.12) that such a coefficient is expressible in terms of lower order coefficients $a_{m,\ell}$, which are already determined, and coefficients a_{m_0,ℓ_0} of order $p+1$ such that $(m_0)_n\le\chi$ and which are supposed to be known by the induction hypothesis on $(m_0)_n$. Therefore, all coefficients a_{m_0,ℓ_0} of order $p+1$ with $(m_0)_n=\chi+1$ can be determined once all coefficients a_{m_0,ℓ_0} up to order $p+1$ with $(m_0)_n\le\chi$ are known. In this manner, all coefficients up to order $p+1$ can be determined once all coefficients of up to order p are known. The claim thus follows.

For a fixed k as in Theorem 3.1 we now consider parabolic polynomials of degree $\le k$. We next introduce a notion which generalizes to the case $k\ge 2$ that in Definition 2.5 above.

Definition 3.4. *We say that $P(x,t) = \sum_{0\le|m|+2\ell\le k} a_{m,\ell}x^m t^\ell$ is an approximating parabolic polynomial of order k for $\frac{v}{u}$ at $(0,0)$ if we have $R(x,t)\equiv 0$ in the representation* (3.10) *above. This is equivalent to the fact that the $a_{m,\ell}$ satisfy* (3.12) *with $d_{q,\kappa}=0$.*

Remark 3.5. *The motivation for Definition 3.4 comes form the fact that, when P is approximating, then from* (3.10) *we obtain $H(uP)=w$, with w of weighted order $k-1+\alpha$. This is*

crucially used in (3.18) *below since it allows us to cancel the term* $r^{k-1+\alpha}$ *from both sides of the equation. We have in fact, see* (3.9)*,* $w(x,t) = (|x|^2 + |t|)^{\frac{k-1+\alpha}{2}} w_1(x,t)$*, where* $||w_1(x,t)|| \leq \delta$*. Note that when* $k = 1$*, and therefore* $P(x,t) = a_0 + \sum_{i=1}^n a_i x_i$*, this notion is equivalent to saying that* $x_n P$ *is caloric, which is in turn equivalent to the condition* $a_n = 0$*. In that case from* (2.15) *we see* $H(uP) \leq C\delta r^{\alpha}$*, which in the case* $k = 1$ *is the* w *of order* $1 - 1 + \alpha = \alpha$*. We also mention that in the case of the variable coefficient operators treated in Section 4 we will no longer impose the condition* $R = 0$ *in the definition of approximating polynomial. This is so because of a nonzero right-hand side.*

The proof of Theorem 3.1 follows the same lines as that of Theorem 2.2 once the following lemma is established.

Lemma 3.6. *Assume that for some* $r \leq 1$ *and* P *an approximating polynomial of order* k *for* $\frac{v}{u}$ *at* $(0,0)$ *with* $||P|| \leq 1$*, one has*

$$||v - uP||_{L^\infty(G\cap Q_r(0,0))} \leq r^{k+1+\alpha}. \tag{3.13}$$

Then, there exists an approximating polynomial $\tilde{P}$ *of order* k *such that for some* $C, \rho > 0$ *universal, we have*

$$||P - \tilde{P}||_{L^\infty(G\cap Q_r(0,0))} \leq C r^{k+\alpha}, \tag{3.14}$$

and

$$||v - u\tilde{P}||_{L^\infty(G\cap Q_{\rho r}(0,0))} \leq (\rho r)^{k+1+\alpha}. \tag{3.15}$$

Proof. With $\tilde{G}$ as in the proof of Lemma 2.6, we define $\tilde{v}(x,t)$ by the equation

$$v(x,t) = u(x,t)P(x,t) + r^{k+1+\alpha}\tilde{v}(\frac{x}{r}, \frac{t}{r^2}), \tag{3.16}$$

and we let

$$\tilde{u}(x,t) = \frac{u(rx, r^2 t)}{r}. \tag{3.17}$$

Since v is a solution of the heat equation we have

$$0 = \Delta v - v_t = (\Delta - D_t)(uP) + r^{k-1+\alpha}(\Delta\tilde{v} - D_t\tilde{v})(\frac{x}{r}, \frac{t}{r^2}).$$

Using the fact that P is an approximating polynomial of order k and (3.10), we conclude

$$r^{k-1+\alpha}(\Delta\tilde{v} - D_t\tilde{v})(\frac{x}{r}, \frac{t}{r^2}) = -w(x,t). \tag{3.18}$$

We emphasize the crucial role played in the latter equation by the property of P being an approximating polynomial, see Remark 3.5 above. By (3.9) we conclude that in $\tilde{G} \cap Q_1(0,0)$ one has

$$(\Delta - D_t)\tilde{v} = h,$$

where

$$h \in H^{k-2+\alpha}(\tilde{G} \cap Q_1(0,0)) \quad \text{and} \quad |h| \leq C\delta.$$

In addition, $\tilde{v}$ vanishes on $\partial\tilde{G} \cap Q_1(0,0)$. Therefore, if we let $\delta \to 0$ along a sequence, we will have by compactness (by using uniform interior $H^{k+1+\alpha}$ estimates and boundary $H^{k+\alpha}$ estimates) that for a subsequence $\delta \to 0$, we have $\tilde{v} = \tilde{v}(\delta) \to v_0$ uniformly on compact subsets, where the limit function v_0 has the properties

$$\begin{cases} \Delta v_0 - D_t v_0 = 0 \ \text{ in } B_1^+ \times (-1, 0], \\ ||v_0||_{L^\infty} \leq 1, \\ v_0 = 0 \text{ on } (\{x_n = 0\} \cap B_1(0)) \times (-1, 0]. \end{cases} \tag{3.19}$$

Proceeding as in the proof of Lemma 2.6 we can find a polynomial Q_0 of degree k such that $x_n Q_0$ solves the heat equation, and

$$||v_0 - x_n Q_0||_{L^\infty(B_\rho^+\times(-\rho^2,0])} \le C\rho^{k+2} \le \frac{1}{4}\rho^{k+1+\alpha}, \tag{3.20}$$

for some $C, \rho > 0$ universal. This follows for instance by odd reflection of v_0 across $x_n = 0$ (which provides again a solution to the heat equation), and by applying Lemma 1.1 in [AV] to the extended function. Then, again for $\delta > 0$ sufficiently small, we have

$$||\tilde{v} - v_0||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))} \le \frac{1}{4}\rho^{k+1+\alpha}. \tag{3.21}$$

Therefore, from (3.20), (3.21) and the triangle inequality we obtain,

$$||\tilde{v} - x_n Q_0||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))} \le \frac{1}{2}\rho^{k+1+\alpha}. \tag{3.22}$$

Now, by using the second inequality in (3.3) we conclude that, if the choice of δ is further restricted in such a way that $C\delta \le \frac{1}{4}\rho^{k+1+\alpha}$, for some C which depends on $||Q_0||$ and hence is universal, then

$$||\tilde{v} - \tilde{u} Q_0||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))} \le \frac{3}{4}\rho^{k+1+\alpha}.$$

Therefore, by rewriting $\tilde{v}$ and $\tilde{u}$ in terms of u and v we have that

$$||v - u\big(P + r^{k+\alpha} Q_0(\frac{\cdot}{r}, \frac{\cdot}{r^2})\big)||_{L^\infty(\tilde{G}\cap Q_{\rho r}(0,0))} \le \frac{3}{4}(\rho r)^{k+1+\alpha}. \tag{3.23}$$

At this point, (3.23) would complete the proof of the lemma if we knew that the parabolic polynomial $P + r^{k+\alpha} Q_0(\frac{\cdot}{r}, \frac{\cdot}{r^2})$ is approximating for $\frac{v}{u}$. Unfortunately, this is not necessarily the case and we need to further modify Q_0 to some other polynomial $\tilde{Q}$, without essentially modifying the estimate (3.23).

Let us write $Q_0(x,t) = \sum_{0\le|q|+2\kappa\le k} b_{q,\kappa} x^q t^\kappa$ and $\tilde{Q}(x,t) = \sum_{0\le|q|+2\kappa\le k} \tilde{b}_{q,\kappa} x^q t^\kappa$. In view of (3.12), and by using the fact that $P(x,t) = \sum_{0\le|m|+2\ell\le k} a_{m,\ell} x^m t^\ell$ is an approximating polynomial for $\frac{v}{u}$ of degree k, we obtain that, in order for

$$\tilde{P}(x,t) \stackrel{def}{=} P(x,t) + r^{k+\alpha}\tilde{Q}(\frac{x}{r}, \frac{t}{r^2}) \tag{3.24}$$

to be an approximating polynomial for $\frac{v}{u}$ of degree k, the coefficients $\tilde{b}_{q,\kappa}$ of $\tilde{Q}$ should satisfy

$$(q_n+1)(q_n+2)\tilde{b}_{q+\overline{n},\kappa} + \sum_{i\ne n}(q_i+1)(q_i+2)\tilde{b}_{q+2\overline{i}-\overline{n},\kappa} - (\kappa+1)\tilde{b}_{q-\overline{n},\kappa+1} + \tilde{c}^{m,\ell}_{q,\kappa}\tilde{b}_{m,\ell} = 0, \tag{3.25}$$

where

$$\tilde{c}^{m,\ell}_{q,\kappa} = r^{|q|+2\kappa+1-(|m|+2\ell)} c^{m,\ell}_{q,\kappa}.$$

The justification for (3.25) is as follows. If we write $P(x,t)+r^{k+\alpha}\tilde{Q}(\frac{x}{r}, \frac{t}{r^2}) = \sum_{0\le|q|+2\kappa\le k} \tilde{a}_{q,\kappa} x^q t^\kappa$, then it is clear that

$$\tilde{a}_{q,\kappa} = a_{q,\kappa} + r^{k+\alpha-|q|-2\kappa}\tilde{b}_{q,\kappa} \tag{3.26}$$

Now, imposing that $P(x,t)+r^{k+\alpha}\tilde{Q}(\frac{x}{r}, \frac{t}{r^2})$ be approximating of order k for $\frac{v}{u}$ at $(0,0)$ is, in view of (3.12) and Definition 3.4, equivalent to

$$(q_n+1)(q_n+2)\tilde{a}_{q+\overline{n},\kappa} + \sum_{i\ne n}(q_i+1)(q_i+2)\tilde{a}_{q+2\overline{i}-\overline{n},\kappa} - (\kappa+1)\tilde{a}_{q-\overline{n},\kappa+1} + c^{m,\ell}_{q,\kappa}\tilde{a}_{m,\ell} = 0. \tag{3.27}$$

Replacing (3.26) in (3.27), and using the fact that P is itself approximating for $\frac{v}{u}$ (see (3.12)), after some elementary computations we recognize that the coefficients $\tilde{b}_{q,\kappa}$ must satisfy

$$r^{k+\alpha}\Bigg[r^{-(|q|+2\kappa+1)}(q_n+1)(q_n+2)\tilde{b}_{q+\overline{n},\kappa}+r^{-(|q|+2\kappa+1)}\sum_{i\neq n}(q_i+1)(q_i+2)\tilde{b}_{q+2\overline{i}-n,\kappa}$$

$$-\, r^{-(|q|+2\kappa+1)}(\kappa+1)\tilde{b}_{q-\overline{n},\kappa+1}+r^{-(|m|+2\ell)}c_{q,\kappa}^{m,\ell}\tilde{b}_{m,\ell}\Bigg]=0.$$

Eliminating $r^{k+\alpha}$ and then multiplying the resulting equation by $r^{|q|+2\kappa+1}$ we finally obtain (3.25). Recalling that $0<r<1$, and that from the defintion of $d_{q,\kappa}$ in (3.11) one has $c_{q,\kappa}^{m,\ell}\neq 0$ only when $|m|+2\ell\leq |q|+2\kappa\leq k-1$, we conclude that

$$|\tilde{c}_{q,\kappa}^{m,\ell}|=r^{|q|+2\kappa+1-(|m|+2\ell)}|c_{q,\kappa}^{m,\ell}|\leq r|c_{q,\kappa}^{m,\ell}|\leq C\delta.$$

Since (3.20) above implies that Q_0 is approximating for $\frac{v_0}{x_n}$, then its coefficients $b_{m,\ell}$ must satisfy

$$(q_n+1)(q_n+2)b_{q+\overline{n},\kappa}+\sum_{i\neq n}(q_i+1)(q_i+2)b_{q+2\overline{i}-\overline{n},\kappa}-(\kappa+1)b_{q-\overline{n},\kappa+1}=0. \tag{3.28}$$

If we now subtract (3.28) from (3.25), we find that the coefficients of $\tilde{Q}-Q_0$ solve a linear system with left-hand side bounded by $C\delta$ and contains unknown coefficients $\tilde{b}_{m,\ell}$ such that $|m|+2\ell$ is less than the sum of the indices of the coefficients of $\tilde{Q}-Q_0$ appearing in the right-hand side. We in fact have

$$\begin{aligned}-\tilde{c}_{q,\kappa}^{m,\ell}\tilde{b}_{m,\ell}&=(q_n+1)(q_n+2)(\tilde{b}_{q+\overline{n},\kappa}-b_{q+\overline{n},\kappa})\\&+\sum_{i\neq n}(q_i+1)(q_i+2)(\tilde{b}_{q+2\overline{i}-\overline{n},\kappa}-b_{q+2\overline{i}-\overline{n},\kappa})-(\kappa+1)(\tilde{b}_{q-\overline{n},\kappa+1}-b_{q-\overline{n},\kappa+1}).\end{aligned} \tag{3.29}$$

The reader should note that the order of the coefficient of any term in the right hand side of the latter equation is $|q|+2\kappa+1>|m|+2\ell$. Again, this is so since $c_{q,\kappa}^{m,\ell}\neq 0$ precisely when $|m|+2\ell\leq|q|+2\kappa$. Consequently, if we set $\tilde{b}_{m,\ell}=b_{m,\ell}$ when $m_n=0$, then in view of the procedure described after (3.12), we can determine all the other coefficients $\tilde{b}_{q,\kappa}$ of $\tilde{Q}$ by induction on the order of the coefficient $|q|+2\kappa$. Morever, since the coefficients of $\tilde{Q}-Q_0$ solve a linear system with left-hand side bounded by $C\delta$, we can further ensure that

$$||\tilde{Q}-Q_0||_{L^\infty(Q_1(0,0))}\leq C\delta.$$

Since

$$||\tilde{Q}||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))}\leq||\tilde{Q}-Q_0||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))}+||Q||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))},$$

we conclude that there exists a universal constant $C>0$ such that

$$||\tilde{Q}||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))}\leq C.$$

Therefore, by (3.22) and by choosing a smaller δ if needed, one can ensure that

$$\begin{aligned}||\tilde{v}-x_n\tilde{Q}||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))}\leq&\ ||\tilde{v}-x_nQ_0||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))}+||x_n(\tilde{Q}-Q_0)||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))}\\\leq&\ \frac{1}{2}\rho^{k+1+\alpha}+C\delta\rho\leq\frac{3}{4}\rho^{k+1+\alpha}.\end{aligned} \tag{3.30}$$

Then, again by the second inequality in (3.3) and by (3.30), we obtain for a smaller choice of δ if needed that

$$\begin{aligned}||\tilde{v}-\tilde{u}\tilde{Q}||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))}&\leq||\tilde{v}-x_n\tilde{Q}||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))}+||\tilde{Q}(\tilde{u}-x_n)||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))}\\&\leq\frac{3}{4}\rho^{k+1+\alpha}+C\delta||\tilde{Q}||_{L^\infty(\tilde{G}\cap Q_\rho(0,0))}\leq\frac{3}{4}\rho^{k+1+\alpha}+C\delta\leq\rho^{k+1+\alpha}.\end{aligned} \tag{3.31}$$

Recalling the definitions (3.16), (3.17) and (3.24), the conclusion (3.15) of the lemma is now obtained as follows

$$\begin{aligned}||v-u\tilde{P}||_{L^\infty(G\cap Q_{\rho r}(0,0))} &= ||uP+r^{k+1+\alpha}\tilde{v}(\frac{\cdot}{r},\frac{\cdot}{r^2})-u(P+r^{k+\alpha}u\tilde{Q}(\frac{\cdot}{r},\frac{\cdot}{r^2})||_{L^\infty(G\cap Q_{\rho r}(0,0))}\\ &= r^{k+1+\alpha}||\tilde{v}(\frac{\cdot}{r},\frac{\cdot}{r^2})-\tilde{u}(\frac{\cdot}{r},\frac{\cdot}{r^2})\tilde{Q}(\frac{\cdot}{r},\frac{\cdot}{r^2})||_{L^\infty(G\cap Q_{\rho r}(0,0))}\\ &= r^{k+1+\alpha}||\tilde{v}-\tilde{u}\tilde{Q}||_{L^\infty(G\cap Q_{\rho}(0,0))} \ \le (\rho r)^{k+1+\alpha},\end{aligned}$$

where in the last inequality we have used (3.31). This completes the proof of the lemma.

□

Proof of Theorem 3.1. The proof of the theorem now follows by iterating Lemma 3.6. To start the process of iteration, we take $P=0$. Multiplying v by a suitable constant, one can ensure that the hypothesis of Lemma 3.6 holds for some r_0 universal. The rest of the proof remains the same as in the case $k=1$, in which Lemma 3.6 is applied iteratively first for r_0, then for $\rho r_0, \rho^2 r_0$, and so on. We finally obtain a limiting polynomial P_0 of degree at most k having the following representation

$$P_0(x,t)=\sum_{i=1}^{\infty}(\rho^{i-1}r_0)^{k+\alpha}\tilde{Q}_i(\frac{x}{\rho^{i-1}r_0},\frac{t}{(\rho^{i-1}r_0)^2}),$$

where $\tilde{Q}_i$ is the polynomial obtained after the i-th application of Lemma 3.6. Furthermore, the following holds

$$||v-uP_0||_{L^\infty(G\cap Q_r(0,0))}\le Cr^{k+1+\alpha}, \qquad r\le r_0. \tag{3.32}$$

For $r>0$ we now consider the functions $\tilde{v},\tilde{u}$ defined, with $P=P_0$, as in the proof of Lemma 3.6. Given that (3.32) holds for any given $r\le r_0$, we can argue as in the case $k=1$ and obtain that

$$|\tilde{v}|\le C\tilde{u} \qquad \text{in } \tilde{G}\cap Q_{1/2}(0,0). \tag{3.33}$$

By rewriting $\tilde{u},\tilde{v}$ in terms of u and v, from (3.33) we obtain for $(x,t)\in G\cap Q_{r/2}(0,0)$

$$|v(x,t)-u(x,t)P_0(x,t)|\le Cu(x,t)r^{k+\alpha}. \tag{3.34}$$

The inequality (3.34) gives for $(x,t)\in G\cap Q_1(0,0)$

$$|\frac{v}{u}(x,t)-P_0(x,t)|\le C(|x|^2+|t|)^{\frac{k+\alpha}{2}}.$$

This implies $H^{k+\alpha}$-regularity at the boundary point $(0,0)$. Finally, by arguing as in the case $k=1$, the identity

$$\frac{v(x,t)}{u(x,t)}=P_0(x,t)+r^{k+\alpha}\frac{\tilde{v}(\frac{x}{r},\frac{t}{r^2})}{\tilde{u}(\frac{x}{r},\frac{t}{r^2})}, \tag{3.35}$$

implies that the $H^{k+\alpha}$ norm of $\frac{v}{u}$ is bounded in regions of the form $G\cap Q_{r/4}(\frac{r}{2}e_n,0)$, i.e., in regions which are away from the boundary of G by a parabolic distance proportional to r. At this point, the conclusion follows similarly to the case $k=1$ by arguing as in the proof of Proposition 4.13 in [CC].

□

4. VARIABLE COEFFICIENTS

In this section we intend to extend Theorem 2.2 and Theorem 3.1 to variable coefficient parabolic operators of the type

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x,t)D_{ij}u + \sum_{i=1}^{n} b_i(x,t)D_iu + c(x,t)u - u_t = 0, \tag{4.1}$$

satisfying appropriate regularity assumptions on the coefficients, and to strong solutions $u \in W^{2,1}_{p,loc}(G)$. Henceforth, we use the notation $\mathrm{Tr}(M)$ for the trace of a matrix M. We can thus write $\sum_{i,j=1}^n a_{ij}(x,t)D_{ij}u = \mathrm{Tr}(A(x,t)D^2u)$, where we have indicated with $D^2u = [D_{ij}u]$ the Hessian matrix of u.

As in the case of the heat equation, in order to better present the ideas we first treat the case $k = 1$ in Section 4.1. Subsequently, in Section 4.2, we treat the case of $k > 1$. In the sequel we will denote with $W^{2,1}_p(G)$ the parabolic Sobolev spaces. For their precise definition we refer the reader to p.155 in [Li]. We will indicate with $W^{2,1}_{p,loc}(G)$ the standard local spaces.

4.1. $H^{1+\alpha}$ regularity.

The assumptions on G are as in the hypothesis of Theorem 2.2. In this section we assume that $A = [a_{ij}] \in H^\alpha(G)$, $b, c \in L^\infty(G)$. The following is our main result.

Theorem 4.1. *Let $p > 1$ and suppose that $u \in W^{2,1}_{p,loc}(G \cap Q_2(0,0))$ be a positive strong solution to* (4.1) *above. Let $v \in W^{2,1}_p(G \cap Q_2(0,0))$ be a strong solution to*

$$Lv = g, \ \ \text{in } G, \tag{4.2}$$

where $g \in H^\alpha(\overline{G \cap Q_2(0,0)})$. Assume that u, v vanish on $\partial_p G \cap Q_2(0,0)$. Furthermore, let u satisfy the normalization condition (2.3)*. Then, one has*

$$||\frac{v}{u}||_{H^{1+\alpha}(G\cap Q_1(0,0))} \leq C(||v||_{L^\infty(G\cap Q_2(0,0))} + ||g||_{H^\alpha(G\cap Q_2(0,0))} + 1), \tag{4.3}$$

for some $C > 0$ universal.

Remark 4.2. *We first note that from the assumptions on the coefficients, the Calderón- Zygmund theory implies that $u \in W^{2,1}_{q,loc}(G)$ for all $1 < q < \infty$, see for instance Proposition 7.14 in* [Li]*. Then, we can invoke Theorem 4.29 in* [Li] *to conclude that v, u are in $H^{1+\alpha}(\overline{G \cap Q_2(0,0)})$. We note that Theorem 4.29 in* [Li] *can be applied to strong solutions in $W^{2,1}_{n+1,loc}$ via approximations by solutions to equations with smooth coefficients and by an application of the comparison principle Theorem 7.1 in* [Li]*. We refer to* [GH] *for the elliptic counterpart of such intermediate Schauder type regularity result.*

After a suitable change of coordinates and parabolic dilation similar to (2.6), we may assume that

$$\begin{cases} A(0,0) = I, \\ f(0) = 0, \quad D'f(0) = 0, \quad ||f||_{C^{1,\alpha}(G\cap Q_2(0,0))} \leq \delta, \\ Du(0,0) = e_n, \quad ||u - x_n||_{H^{1+\alpha}(G\cap Q_2(0,0))} \leq \delta, \\ \max\left\{ [A]_{\alpha, G\cap Q_2(0,0)}, ||g||_{H^\alpha(G\cap Q_2(0,0))}, ||b,c||_{L^\infty(G\cap Q_2(0,0))} \right\} \leq \delta. \end{cases} \tag{4.4}$$

Here, $[A]_\alpha$ indicates the α-Hölder seminorm of $[a_{ij}]$, see p. 46 in [Li]. More precisely, first by a suitable change of coordinates, we can ensure that $A(0,0) = I$. Then, by letting

$$u_{r_0}(x,t) = \frac{u(r_0x, r_0^2t)}{r_0} \qquad v_{r_0}(x,t) = \frac{v(r_0x, r_0^2t)}{r_0}, \tag{4.5}$$

as in (2.6), we have that u_{r_0}, v_{r_0} solve in G^{r_0} (same definition as in Section 2)

$$L_0 u_{r_0} = 0, \quad L_0 v_{r_0} = g_0,$$

where

$$L_0 w = \mathrm{Tr}(A_0 D^2 w) + < b_0, Dw > + c_0 w - w_t,$$

and

$$A_0(x,t) = A(r_0 x, r_0^2 t),\ b_0(x,t) = r_0 b(r_0 x, r_0^2 t),\ c_0 = r_0^2 c(r_0 x, r_0^2 t),\ g_0(x,t) = r_0 g(r_0 x, r_0^2 t).$$

Therefore, if r_0 is suitably chosen depending on δ, (4.4) can be ensured. As before, by abuse of notation, we keep calling $u_{r_0} = u$, $G^{r_0} = G$ and so on. Moreover, as in Section 2, δ will be determined later.

We now introduce the relevant notion of approximating function with respect to L and g, where g is as in Theorem 4.1.

Definition 4.3. *We say that* $P(x) = a_0 + \sum_{i=1}^n a_i x_i$ *is an approximating affine function at* $(0,0)$ *for* $\frac{v}{u}$ *with respect to* L, g *if* $2a_n = g(0,0)$.

With this notion the corresponding statement of Lemma 2.6 remains the same, but its proof needs to be slightly modified.

Lemma 4.4. *Let* u, v *be as in Theorem 4.1. Assume that for some* $r \leq 1$ *and* $P(x) = a_0 + \sum_{i=1}^n a_i x_i$ *an approximating affine function at* $(0,0)$ *for* $\frac{v}{u}$ *(with respect to* L *and* g*) with* $|a_i| \leq 1$*, one has*

$$||v - uP||_{L^\infty(G \cap Q_r(0,0))} \leq r^{2+\alpha}. \tag{4.6}$$

Then, there exists an approximating affine function $\tilde{P}$ *such that for some* $C, \rho > 0$ *universal, we have*

$$||P - \tilde{P}||_{L^\infty(G \cap Q_r(0,0))} \leq C r^{1+\alpha}, \tag{4.7}$$

and

$$||v - u\tilde{P}||_{L^\infty(G \cap Q_{\rho r}(0,0))} \leq (\rho r)^{2+\alpha}. \tag{4.8}$$

Proof. We point out the essential modifications in the proof of Lemma 2.6 in the present context. Let $\tilde{v}$ and $\tilde{u}$ be as in the proof of Lemma 2.6. Then, one has

$$g = Lv = L(uP) + r^\alpha \tilde{L}\tilde{v}(x/r, t/r^2), \tag{4.9}$$

where

$$\tilde{L}\tilde{v} = Tr(\tilde{A} D^2 \tilde{v}) + r < \tilde{b}, D\tilde{v} > + r^2 \tilde{c}\tilde{v} - \tilde{v}_t, \tag{4.10}$$

and

$$\tilde{A} = A(rx, r^2 t),\ \tilde{b} = b(rx, r^2 t),\ \tilde{c} = c(rx, r^2 t), \quad (x,t) \in \tilde{G}. \tag{4.11}$$

Since $Lu = 0$, one has

$$L(uP) = 2 < ADu, DP > + u < b, DP > .$$

From (4.4) we have for all $(x,t) \in G \cap Q_r(0,0)$

$$\begin{cases} A(x,t) = I + \delta r^\alpha M(x,t), \\ Du(x,t) = e_n + \delta r^\alpha w(x,t), \\ g(x,t) = g(0,0) + \delta r^\alpha w_1(x,t), \\ |u(x,t)| \leq C\delta r, \end{cases} \tag{4.12}$$

where M, w, w_1 are bounded. Therefore, using (4.12) and (4.4) and the fact that P is approximating for $\frac{v}{u}$, we obtain for some bounded function $K(x,t)$,

$$|L(uP) - g| = |2 < ADu, DP > + u < b, DP > - g| = |g(0,0) - g(x,t) + \delta r^\alpha K(x,t)| \leq K_1 \delta r^\alpha.$$

Combined with (4.9) this estimate gives

$$|\tilde{L}\tilde{v}| \leq C\delta \quad \text{in} \quad \tilde{G} \cap Q_1(0,0).$$

We also note that, with $\tilde{u}$ as in (2.13), we have

$$\tilde{L}\tilde{u} = 0.$$

Letting $\delta \to 0$, from (4.4) we obtain that up to a subsequence $\delta \to 0$, $\tilde{v} = \tilde{v}(\delta) \to v_0$, which solves (2.17). Note that, unlike the case of the heat equation, we do not presently have uniform interior $H^{2+\alpha}$ estimates for $\tilde{v}$. Nevertheless, because of $H^{1+\alpha}$ estimates for $\tilde{v} = \tilde{v}(\delta)$ up to $\partial_p \tilde{G} \cap Q_1(0,0)$ independent of δ, by applying Theorem 6.1 in [CKS] we can ensure that v_0 is a L^p viscosity solution of the heat equation in the sense of [CKS]. The regularity theory for viscosity solutions now ensures that v_0 is a classical solution of the heat equation. As in the proof of Lemma 2.6 we have that (2.18)-(2.23) holds for $Q_0(x) = \sum_{i=1}^n q_i x_i + q_0$, with $q_n = 0$. Then, as in the case of heat equation the conclusion of the lemma follows with $\tilde{P} = P + r^{1+\alpha} Q_0(\frac{\cdot}{r})$. Note that, if we let $P(x) = \sum_{i=1}^n a_i x_i + a_0$ and $\tilde{P}(x) = \sum_{i=1}^n \tilde{a}_i x_i + \tilde{a}_0$, then since P is an approximating affine function for L and g we have $a_n = \frac{g(0,0)}{2}$. Since $q_n = 0$, we have

$$\tilde{a}_n = a_n + r^\alpha q_n = \frac{g(0,0)}{2}.$$

This shows that also $\tilde{P}$ is an approximating affine function at $(0,0)$ for $\frac{v}{u}$ with respect to L and g. From this fact, the verification of (4.27) above follows as that of (2.11) in Lemma 2.6.

□

Proof of Theorem 4.1. We repeatedly apply Lemma 4.4. To start the process we first take $P(x) = \frac{g(0,0)}{2} x_n$. Then, (4.6) holds for some universal $r = r_0$ when v, g and P are multiplied by suitable constants. As before, by iterating Lemma 4.4 with $r = r_0, \rho r_0, \rho^2 r_0$ and so on, we obtain a limiting affine function P_0 such that (2.33) holds. We note that in this case, P_0 has the following explicit representation

$$P_0(x) = P(x) + \sum_{i=1}^{\infty} (\rho^{i-1} r_0)^{1+\alpha} Q_i\left(\frac{x}{\rho^{i-1} r_0}\right) \tag{4.13}$$

where $P(x) = \frac{g(0,0)}{2} x_n$, and $Q_i(x)$ is the affine function obtained in the i-th iteration of Lemma 4.4. The rest of the proof remains the same as that for the heat equation.

□

4.2. $H^{k+\alpha}$ **regularity for** $k \geq 2$. In what follows the assumptions on G are as in Section 3. We have the following higher-regularity result for variable coefficient operators. We note that in the next result we do not assume that u and v are strong solutions as in Theorem 4.1 since by the regularity theory one infers that both u and v are classical solutions.

Theorem 4.5. *Let u and v be (classical) solutions in $G \cap Q_2(0,0)$ of the equations*

$$Lu = \text{Tr}(AD^2 u) + \langle b, Du \rangle + cu - u_t = 0, \tag{4.14}$$

and

$$Lv = g, \tag{4.15}$$

where $A, g \in H^{k-1+\alpha}(\overline{G \cap Q_2(0,0)})$, $b, c \in H^{k-2+\alpha}(\overline{G \cap Q_2(0,0)})$. Assume that u, v vanish on $\partial_p G \cap Q_2(0,0)$. Also, let $u > 0$ in $G \cap Q_2(0,0)$, and assume furthermore that it satisfy the normalization condition (2.3)*. Then, one has*

$$||\frac{v}{u}||_{H^{k+\alpha}(G \cap Q_1(0,0))} \leq C(||v||_{L^\infty(G \cap Q_2(0,0))} + ||g||_{H^{k-1+\alpha}(G \cap Q_2(0,0))} + 1), \tag{4.16}$$

for some $C > 0$ universal.

As before, from the Schauder theory as in Chapters 4 and 5 in [Li], we have that $u, v \in H^{k+\alpha}(\overline{G \cap Q_{3/2}(0,0)})$. By a suitable change of coordinates and parabolic dilations similar to (2.6), we can assume that

$$
\begin{cases} A(0,0) = I, \quad ||A - I||_{H^{k-1+\alpha}(G\cap Q_2(0,0))} \leq \delta, \\ f(0) = 0, \quad D'f(0) = 0, \quad ||f||_{H^{k+\alpha}(G\cap Q_2(0,0))} \leq \delta, \\ Du(0,0) = e_n, \quad ||u - x_n||_{H^{k+\alpha}(G\cap Q_2(0,0))} \leq \delta, \\ \max\left\{ ||g||_{H^{k-1+\alpha}(G\cap Q_2(0,0))}, ||b, c||_{H^{k-2+\alpha}((G\cap Q_2(0,0))} \right\} \leq \delta, \end{cases} \tag{4.17}
$$

where $0 < \delta < 1$ is to be chosen appropriately later.

Similarly to what was done in the proof of Theorem 3.1 above, we need to compute $L(uP)$, where $P(x,t) = \sum_{0\leq|m|+2\ell\leq k} a_{m,\ell} x^m t^\ell$ is a parabolic polynomial of degree k. Since by (4.14) we have $Lu = 0$, we obtain

$$
L(uP) = uLP + 2 < ADu, DP > + < b, DP > . \tag{4.18}
$$

By the linearity of L we are thus led to understand (4.18) when $P(x,t) = x^m t^\ell$. In such case, from (4.17) it follows that

$$
\begin{aligned} L(uP) &= m_n(m_n+1)x^{m-\overline{n}}t^\ell + \sum_{i\neq n} m_i(m_i-1)x^{m-2\overline{i}+\overline{n}}t^\ell - \ell x^{m+\overline{n}}t^{\ell-1} \\ &+ \sum_{|m|+2\ell\leq|q|+2\kappa\leq k-1} c^{m,\ell}_{q,\kappa} x^q t^\kappa + w_{m,\ell}, \end{aligned} \tag{4.19}
$$

where because of the representation (3.4) which is valid for u, (4.17) and (4.18), we have that

$$
|c^{m,\ell}_{q,\kappa}| \leq C\delta, |w_{m,\ell}| \leq C\delta((|x|^2+|t|)^{\frac{k-1+\alpha}{2}}), \text{ and } ||w_{m,\ell}||_{H^{k-2+\alpha}(G\cap Q_r)} \leq C\delta r. \tag{4.20}
$$

In (4.19), we used the fact that although the term $u < b, DP > \in H^{k-2+\alpha}(G \cap Q_2(0,0))$, but nevertheless (4.19) and (4.20) can be justified as follows. We first note (see also (3.7) above), that because of (4.17), we have

$$
u(x,t) = x_n + P_1(x,t) + w_1(x,t), \tag{4.21}
$$

where P_1 is a polynomial such that $2 \leq \deg(P_1) \leq k$ and

$$
||P_1|| \leq C\delta, \qquad |w_1(x,t)| \leq C\delta(|x|^2+|t|)^{\frac{k+\alpha}{2}}.
$$

We furthermore note that, since $b \in H^{k-2+\alpha}(\overline{G \cap Q_2(0,0)})$, we can write

$$
b(x,t) = P_b(x,t) + b_1(x,t), \tag{4.22}
$$

where P_b is a vector field in $\mathbb{R}^n$ each of whose components are polynomials of degree at most $k-2$. Moreover, because of (4.17) the following holds

$$
||P_b|| \leq C\delta, \qquad |b_1(x,t)| \leq C\delta(|x|^2+|t|)^{\frac{k-2+\alpha}{2}}.
$$

Therefore, from (4.21) and (4.22) it follows that

$$
u < b, DP >= P_{u,b}(x,t) + w_{u,b}(x,t),
$$

where $P_{u,b}(x,t)$ is a polynomial of degree at most $k-1$ such that

$$
||P_{u,b}|| \leq C\delta, \; |w_{u,b}(x,t)| \leq C\delta(|x|^2+|t|)^{\frac{k-1+\alpha}{2}}.
$$

We also observe that in (4.20) we have that $c^{m,\ell}_{q,\kappa} \neq 0$ only when $|m| + 2\ell \leq |q| + 2\kappa \leq k-1$, and that, furthermore, the coefficient does depend on A, u and b. Now for a general polynomial of the form $\sum_{0\leq|m|+2\ell\leq k} a_{m,\ell} x^m t^\ell$ one has

$$
L(uP) = R(x,t) + \sum a_{m,\ell} w_{m,\ell}(x,t), \tag{4.23}
$$

where

$$R(x,t) = \sum_{0\le |q|+2\kappa\le k-1} d_{q,\kappa} x^q t^\kappa, \tag{4.24}$$

and the coefficients $a_{m,\ell}$ and $d_{q,\kappa}$ of P, and $c_{q,\kappa}^{m,\ell}$ satisfy (3.12) as in the case of heat equation.

W e next introduce the appropriate notion of approximating polynomial in the present context.

Definition 4.6. *We say that a parabolic polynomial P of degree $\le k$ is approximating of order k at $(0,0)$ for $\frac{v}{u}$ with respect to L and g if the coefficients $d_{\ell,m}$ of $R(x,t)$ in the representation (4.23), (4.24) above coincide with the coefficients of the Taylor polynomial of order $k-1$ for g at $(0,0)$.*

With this notion in place, we now state the analogue of Lemma 3.6.

Lemma 4.7. *Let u, v be as in Theorem 4.5. Assume that for some $r \le 1$ and P an approximating polynomial of order k for $\frac{v}{u}$ at $(0,0)$ with respect to L and g, with $||P|| \le 1$, one has*

$$||v - uP||_{L^\infty(G\cap Q_r(0,0))} \le r^{k+1+\alpha}. \tag{4.25}$$

Then, there exists an approximating polynomial $\tilde{P}$ of order k such that for some $C, \rho > 0$ universal, we have

$$||P - \tilde{P}||_{L^\infty(G\cap Q_r(0,0))} \le C r^{k+\alpha}, \tag{4.26}$$

and

$$||v - u\tilde{P}||_{L^\infty(G\cap Q_{\rho r}(0,0))} \le (\rho r)^{k+1+\alpha}. \tag{4.27}$$

Proof. The proof of Lemma 4.7 follows by arguing as in that of Lemma 3.6. We define

$$v = uP + r^{k+1+\alpha}\tilde{v}(\frac{\cdot}{r}, \frac{\cdot}{r^2}),$$

where P satisfies the hypothesis of the lemma. Since $g \in H^{k-1+\alpha}(\overline{G\cap Q_2(0,0)})$, from the bounds in (4.20) we have for all $(x,t) \in G \cap Q_r(0,0)$,

$$|g(x,t) - P_g(x,t)| \le C\delta(|x|^2 + |t|)^{\frac{k-1+\alpha}{2}}, \tag{4.28}$$

where P_g is a polynomial of degree at most $k-1$.

We note that since P is approximating for $\frac{v}{u}$ at $(0,0)$ with respect to L and g, we have that $P_g(x,t) = R(x,t)$. Therefore, from (4.19) and the bounds in (4.20) we obtain in $G \cap Q_r(0,0)$

$$|r^{k-1+\alpha}\tilde{L}\tilde{v}(\frac{x}{r}, \frac{t}{r^2})| = |L(uP) - g| \le C\delta r^{k-1+\alpha}, \tag{4.29}$$

where $\tilde{L}$ is as in (4.10). The estimate (4.29) implies

$$|\tilde{L}\tilde{v}| \le C\delta, \tag{4.30}$$

where $\tilde{L}$ is as in (4.10). As a consequence, for a subsequence $\delta \to 0$ we have $\tilde{v} = \tilde{v}(\delta) \to v_0$, where v_0 is as in (3.19). Now, similarly to the proof of Lemma 3.6 there exists Q_0 such that (3.20)-(3.23) holds. Moreover, as in the case of heat equation, the polynomial $P(x,t) + r^{k+\alpha}Q_0(\frac{x}{r}, \frac{t}{r^2})$ need not be approximating for $\frac{v}{u}$ with respect to L and g. Therefore as before, we modify Q_0 to $\tilde{Q}$ such that $P(x,t) + r^{k+\alpha}\tilde{Q}(\frac{x}{r}, \frac{t}{r^2})$ is an approximating polynomial of order k for $\frac{v}{u}$ at $(0,0)$ with respect to L and g. Since P is already an approximating polynomial for $\frac{v}{u}$ the coefficients of $\tilde{Q}$ should satisfy (3.25) similarly to the situation of the heat equation. The only difference in the present case being that in the analogue of (3.25) the coefficients $\tilde{c}_{q,\kappa}^{m,\ell}$ would additionally depend on A and b, besides u. The rest of the arguments remain the same as in the proof of Lemma 3.6, and the desired conclusion follows.

□

Proof of Theorem 4.5. As previously, it follows by applying Lemma 4.7 repeatedly. In this case, in order to start the process of iteration we determine an approximating polynomial $P = \sum_{m,\ell} a_{m,\ell} x^m t^\ell$ from (3.12) where $d_{q,\kappa}$'s are determined by the Taylor polynomial $P_g(x,t)$ of order $k-1$ for g. In view of the procedure described after (3.12), such a polynomial can be determined. Then, by multiplying v and P by a suitable constant, the hypothesis of Lemma 4.7 holds for some small enough universal r_0. Therefore, by applying the Lemma 4.7 iteratively with $r = r_0, \rho r_0, \rho^2 r_0$ and so on, we obtain a limiting polynomial P_0 which has the following representation

$$P_0(x,t) = P(x,t) + \sum_{i=1}^{\infty} (\rho^{i-1} r_0)^{k+\alpha} \tilde{Q}_i\left(\frac{x}{\rho^{i-1} r_0}, \frac{t}{(\rho^{i-1} r_0)^2}\right), \tag{4.31}$$

where P is the above polynomial which is determined before the first step of the iteration and $\tilde{Q}_i$ are the polynomials determined after the i-th application of Lemma 4.7. Moreover, with such a P_0, we have that (3.32) holds. The rest of the proof remains the same as that for the heat equation.

□

5. Application to the parabolic obstacle problem

As mentioned in the introduction, we close the paper with an application of Theorem 3.1 to the parabolic obstacle. For a measurable set $E \subset \mathbb{R}^{n+1}$, we indicate with χ_Ω its indicator function. We consider the following problem studied in [CPS].

Problem 5.1. *Given a domain $D \subset \mathbb{R}^n \times \mathbb{R}$, consider a function $u(x,t)$ defined in D such that u, Du are continuous, and define the* coincidence set *as*

$$\Lambda = \{(x,t) \in D \mid u(x,t) = Du(x,t) = 0\}.$$

With $\Omega = D \setminus \Lambda$, suppose that u solves the equation

$$\Delta u - u_t = \chi_\Omega.$$

The free boundary *is defined as*

$$\Gamma = \Gamma(u) = \partial\Omega \cap D.$$

With this setup, we mention a corollary of Theorem 3.1.

Corollary 5.2. *Let $(x_0, t_0) \in \Gamma$ be a point as in Theorem 13.1 and Lemma 13.3 in* [CPS]. *Then, $\partial\Omega \cap Q_{1/4}(x_0,t_0)$ is C^∞.*

Proof. By the $C_x^{1,1}$ regularity of u as in Section 4 in [CPS], we have that the spatial derivatives $D_i u$ vanish on $\partial\Omega$ for all $i = 1,..n$. Moreover, Lemma 13.3 in [CPS] implies that $D_t u$ also vanishes continuously at the free boundary $\Gamma \cap Q_{1/4}(x_0,t_0)$ near (x_0,t_0). Now, Theorem 14.1 in [CPS] implies that $Q_{1/4}(x_0,t_0) \cap \Gamma$ is $C^{1,\alpha}$ regular which follows by an application of the boundary Harnack inequality as in [ACS]. Moreover, in the proof of Theorem 14.1 in [CPS] it is evident that, without loss of generality, one can assume that $\Gamma \cap Q_{\rho/4}(x_0,t_0) = \{(x,t) \mid x_n = f(x',t)\}$ and that the following holds,

$$u_n(x_0 + \frac{3}{16} e_n, t_0 - (\rho/16)^2) \geq c_0, \tag{5.1}$$

for some $c_0, \rho > 0$ universal. Moreover, ρ can be chosen in such a way that the point $(x_0 + \frac{3}{16} e_n, t_0 - (\rho/16)^2)$ is at a parabolic distance from Γ bounded from below by a universal constant C_0. Now

$$u(x', f(x',t), t) = 0. \tag{5.2}$$

Therefore, by differentiating the equation (5.2) with respect to the variables $x_1, ...x_{n-1}, t$, we obtain that

$$\frac{D_i u}{D_n u} = D_i f, \ \frac{D_t u}{D_n u} = D_t f. \tag{5.3}$$

Since (5.1) is a scaled version of the normalization (2.3), this implies that if we take $v = D_i u$ and $u = D_n u$ in Theorem 2.2, we obtain from (5.3) that $D' f \in H^{1+\alpha}$. Similarly, with $v = D_t u$ and $u = D_n u$, by application of Theorem 2.2 we find that $D_t f \in H^{1+\alpha}$. This implies that $f \in H^{2+\alpha}$, i.e., the free boundary is $H^{2+\alpha}$ regular. We now proceed inductively as follows. Suppose we know that f and hence the free boundary is in $H^{k+\alpha}$ for some $k \geq 2$. Then, by applying Theorem 3.1 to $v = D_i u$ and $u = D_n u$, we obtain from (5.3) that $D' f \in H^{k+\alpha}$. Similarly, with $v = D_t u$ and $u = D_n u$, we find that $D_t f \in H^{k+\alpha}$. This clearly implies that $f \in H^{k+1+\alpha}$ and hence the free boundary $\Gamma \cap Q_{1/4}(x_0, t_0)$ is $H^{k+1+\alpha}$. Therefore, we can repeatedly apply Theorem 3.1 to conclude that $\Gamma \cap Q_{1/4}(x_0, t_0)$ is smooth.

□

Remark 5.3. *As mentioned before in the introduction, one can in fact establish space-like real analyticity of the free boundary by employing the hodograph transform in* [CPS] *(see Theorem 15.1 in* [CPS]*). Nevertheless, similarly to the elliptic case, the proof of Corollary 5.2 provides a new perspective in the study of parabolic free boundary problems. It remains an interesting question to see if one can establish an analogue of Theorem 3.1 and Corollary 5.2 in the thin parabolic obstacle problem studied in* [DGPT] *where the use of the hodograph transformation does not appear feasible.*

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